International Journal of Theoretical Physics, Vol. 8, No. 3 (1973), pp. 171–178

On the Trace of the Product of Pauli Matrices Occurring as $a_r = a_{r_i}\sigma_i + ia_{r_4}$ and that of the Product of Dirac Matrices, and the Interconnection between them

S. SARKAR

Department of Theoretical Physics, Indian Association for the Cultivation of Science, Jadavpur, Calcutta-32, India

Received: 20 *September* 1972

Abstract

In this paper we have evaluated $\sigma_i u \sigma_i$, $\sigma_i \sigma_j u \sigma_i \sigma_j$, $\mathrm{Tr}(\sigma_i u)\mathrm{Tr}(\sigma_i v)$, $\mathrm{Tr}(\sigma_i \sigma_j u)\mathrm{Tr}(\sigma_i \sigma_j v)$ and $\mathrm{Tr}(u)$ where u and v involve Pauli matrices σ_i and the 2×2 unit matrix in the form of products of elements of the type $a_r = a_{r_i}\sigma_i + ia_{r_4}$, with the help of the results of the trace calculation involving Dirac matrices. We have evaluated $\gamma_\nu U \gamma_\nu$, $\gamma_\mu S \gamma_\mu$, $\gamma_\mu \gamma_\nu U \gamma_\mu \gamma_\nu$, $\mathrm{Tr}(\gamma_\mu S)\mathrm{Tr}(\gamma_\mu S')$, $\mathrm{Tr}(\gamma_\mu \gamma_\nu U)\mathrm{Tr}(\gamma_\mu \gamma_\nu V)$, $\mathrm{Tr}(\gamma_5 U)\mathrm{Tr}(\gamma_5 V)$, $\mathrm{Tr}(\gamma_5 U)$ and $\mathrm{Tr}(U)$. Here U, V are products of an even number of elements and S, S' are products of an odd number of elements of the type A_r $(= A_{r_\mu}\gamma_\mu)$. We have also dealt with the cases in which the dummy suffixes i and μ occurring in some of the above expressions are replaced by 'a' which assume any specific value instead of implying a summation. We have considered also the evaluation of the above-mentioned traces when the term, $1 \pm \gamma_5$, occurs within the trace brackets; this is required in the calculation of the traces involving σ_i and the unit 2×2 matrix. It has been shown that the problem of the trace calculation involving Dirac matrices can be reduced to one involving three Pauli matrices σ_i and the unit 2×2 matrix.

Introduction

In a previous paper (Sarkar, 1971) we have dealt with the trace of the product of an arbitrary number of elements a_r which involve Pauli matrices in the manner $a_r = a_{r_i}\sigma_i$ (summation over repeated suffix is usually implied). In this paper we consider the same problem when $a_r = a_{r_i}\sigma_i + ia_{r_4} = a_{r_\mu}\sigma_\mu$ (where we write $\sigma_4 = i$). In the notation used in this paper, Greek suffixes μ, ν, . . ., etc., stand for any number from 1 to 4, while Latin suffixes i, j, . . ., etc., take any value from 1 to 3. It is shown in this paper that the above problem involving elements like a_r can be related to an equivalent problem involving A_r $(= A_{r_\mu}\gamma_\mu)$ and a term $1 \pm \gamma_5$. γ_μ are Dirac matrices. Further, the trace of the product of Dirac matrices A_r $(= A_{r_\mu}\gamma_\mu)$ and the same containing one γ_5 matrix can also be related to one involving four 2×2 matrices, namely, σ_i and 1. Now A_r considered here does not involve any 4×4 unit matrix as we find in the term $p_\mu \gamma_\mu + im$ occurring in the perturbation calculation for a particle with mass m and four momenta p_μ. We may infer that the trace calculation can be reduced to one which involves four 2×2 matrices, σ_i and 1 when we deal with the scattering problem for massless fermions. We may point out that the term $1 \pm \gamma_5$ occurs when we deal with the weak interaction problem involving leptons. The term $1 \pm \gamma_5$ is related to the helicity or the spin projection operator for a massless particle. In this paper we also give results for the trace of Dirac matrices when the term $1 \pm \gamma_5$ occurs within the trace brackets. This result may be useful in the solution of weak interaction problems and other problems involving extremely relativistic longitudinally polarised particles.

We have first shown that any even string which is the product of an even number of elements like A_r $(= A_{r_\mu}\gamma_\mu)$ can be written in a form in which the new elements consist of two 2×2 submatrices along the diagonal. The 2×2 submatrices involve the Pauli matrices σ_i and the 2×2 unit matrix. Throughout this paper we have used the abbreviated notation of the trace bracket, $\mathrm{Tr}(U) = (U)$. We have evaluated $\gamma_\mu U \gamma_\mu$, $\gamma_\mu \gamma_\nu U \gamma_\mu \gamma_\nu$, $(\gamma_\mu \gamma_\nu U)(\gamma_\mu \gamma_\nu V)$, $\gamma_\mu S \gamma_\mu$ and $(\gamma_\mu S)(\gamma_\mu S')$ where U, V are even strings and S, S' are odd strings. $\gamma_\mu S \gamma_\mu$ and $(\gamma_\mu S)(\gamma_\mu S')$ were also evaluated by Chisholm (1963) using a different method. We have further given an evaluation of $(\gamma_5 U)(\gamma_5 V)$ in a form which does not involve the γ_5 matrix. Application of this result yields a reduction formula for an efficient evaluation of $(\gamma_5 U)$, (U) and $([1 + \gamma_5]U)$.

In this paper we use the convention that capital letters S, S', U and V denote strings involving Dirac matrices and small letters u, v stand for the same involving Pauli matrices σ_i and 1. With the help of the results for the trace calculation involving Dirac matrices we have derived the formulae for the evaluation of $\sigma_\mu u \sigma_\mu$, $\sigma_\mu \sigma_\nu u \sigma_\mu \sigma_\nu$, $(\sigma_\mu u)(\sigma_\mu v)$, $(\sigma_\mu \sigma_\nu u)(\sigma_\mu \sigma_\nu v)$, (u). Some reduction formulae for $\gamma_a S \gamma_a$, $(\gamma_a S)(\gamma_a S')$, $\sigma_a u \sigma_a$ and $(\sigma_a u)(\sigma_a v)$ are obtained in this paper where the particular repeated suffix 'a' does not imply any summation and can assume any specific value permitted for the suffix. In this connection we may point out that Chisholm (1966) has evaluated the sum $\sum_{r=1}^{3} \sigma_r \sigma_a \sigma_b \ldots \sigma_d \sigma_r$ and $\sum_{r=1}^{3} \ldots \sigma_r \ldots T_r(\sigma_r \sigma_a \sigma_b \ldots \sigma_d)$.

Calculation

Let us define U to be an even string of Dirac matrices of the following type

$$U = A_1, A_2, \ldots A_n = \prod_{p=1}^{n} A_r \tag{1}$$

where n is even and

$$A_r = A_{r_i}\gamma_i + A_{r_4}\gamma_4 = A_{r_\mu}\gamma_\mu \tag{2}$$

We can write

$$U = [iA_{1i}\gamma_i\gamma_4\gamma_5 + i\gamma_5 A_{1,4}][-i\gamma_5\gamma_4][iA_{2i}\gamma_i\gamma_4\gamma_5 + i\gamma_5 A_{2,4}] \times [-i\gamma_5\gamma_4] \ldots [iA_{n_i}\gamma_i\gamma_4\gamma_5 + i\gamma_5 A_{n_4}][-i\gamma_5\gamma_4] \tag{3}$$

$$= \prod_{r=1}^{n} [iA_{r_i}\gamma_i\gamma_4\gamma_5 - i(-1)^r \gamma_5 A_{r_4}] \tag{4}$$

Then we have

$$(U[1 \pm \gamma_5]) = \left(\prod_{r=1}^{n} [iA_{r_i}\gamma_i\gamma_4\gamma_5 \mp i(-1)^r A_{r_4}] [1+\gamma_5] \right) \tag{5}$$

We choose the following representation for Dirac matrices

$$\gamma_k = \begin{pmatrix} 0 & -i\sigma_k \\ i\sigma_k & 0 \end{pmatrix}, \qquad \gamma_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then we get

$$i\gamma_i\gamma_4\gamma_5 = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \tag{6}$$

With the help of relations (5) and (6) we can express the trace involving four 2 × 2 matrices, viz., σ_i and the 2 × 2 unit matrix in terms of the trace involving Dirac matrices in the following manner.

$$(u) = \left(\prod_{r=1}^{n} [a_{r_i}\sigma_i + ia_{r_4}] \right) = \left(\prod_{r=1}^{n} a_{r_\mu}\sigma_\mu \right) \tag{7}$$

$$= \tfrac{1}{2}(U[1+\gamma_5]) \tag{8}$$

where

$$\sigma_4 = i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{9}$$

a_{r_μ} and A_{r_μ} are interrelated in the following manner

$$a_{r_i} = A_{r_i}, \qquad a_{r_4} = -(-1)^r A_{r_4} \tag{10}$$

With the help of equations (7), (8) and (10) and the relation

$$(U) = (U_R) \tag{11}$$

we have

$$(u_R) = \tfrac{1}{2}(\bar{U}_R[1+\gamma_5]) \tag{12}$$

$$= \tfrac{1}{2}(\gamma_4 U_R \gamma_4 \, [1+\gamma_5]) = \tfrac{1}{2}(\gamma_4 U \gamma_4 [1+\gamma_5]) = \tfrac{1}{2}(\bar{U}[1+\gamma_5]) \tag{13}$$

and

$$(\bar{u}) = \tfrac{1}{2}(\bar{U}[1+\gamma_5]) = \tfrac{1}{2}(U[1-\gamma_5]) \tag{14}$$

From equations (13) and (14) we get

$$(\bar{u}) = (u_R) \tag{15}$$

Where the suffix R of U_R and u_R denotes that we have to deal with a reversed string, formed by writing the matrices in the reverse order, e.g.,

$$U_R = A_n, A_{n-1}, \ldots A_1 \tag{16}$$

$$u_R = a_n, a_{n-1}, \ldots a_1 \tag{17}$$

In $\bar{U}$ and $\bar{u}$ the relevant quantities $\bar{A}_{r_\mu}$ and $\bar{a}_{r_\mu}$ are given by

$$\bar{A}_{r_i} = A_{r_i}, \qquad \bar{A}_{r_4} = -A_{r_4} \tag{18}$$

$$\bar{a}_{r_i} = a_{r_i}, \qquad \bar{a}_{r_4} = -a_{r_4} \tag{19}$$

So that the notation of 'bar' implies that the sign of the fourth component is to be reversed.

Equations (8), (14) and (15) yield the following relations,

$$(U) = (u) + (\bar{u}) = (u) + (u_R) \tag{20}$$

$$(\gamma_5 U) = (u) - (\bar{u}) = (u) - (u_R) \tag{21}$$

We know that six elements of the type $\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu$ are the same as six independent elements of the type $\gamma_\mu \gamma_1 - \gamma_1 \gamma_\mu$ and $\gamma_5[\gamma_\mu \gamma_1 - \gamma_1 \gamma_\mu]$. Here γ_1 may be replaced by any one of the four Dirac matrices and to indicate this we replace γ_1 by γ_a. Suffixes μ and ν are assumed to take all values from 1 to 4. Then we have the following obvious relations.

$$\tfrac{1}{2}[\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu]\, U [\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu] = [\gamma_\mu \gamma_a - \gamma_a \gamma_\mu]\, U [\gamma_\mu \gamma_a - \gamma_a \gamma_\mu] + \gamma_5[\gamma_\mu \gamma_a - \gamma_a \gamma_\mu]\, U \gamma_5 [\gamma_\mu \gamma_a - \gamma_a \gamma_\mu] \tag{22}$$

$$= \tfrac{1}{4}[\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu]\, U [\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu] + \tfrac{1}{4}\gamma_5[\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu]\, U \gamma_5 [\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu] \tag{23}$$

When U is an even string equation (22) provides us with the following relation

$$\gamma_\mu \gamma_\nu U \gamma_\mu \gamma_\nu = 4 \gamma_\mu \gamma_a U \gamma_\mu \gamma_a \tag{24}$$

From equation (24) we get

$$\gamma_\mu \gamma_\nu U \gamma_\mu \gamma_\nu = 4 \gamma_\mu \gamma_a \gamma_\lambda A_{1\lambda} A_2 \ldots A_n \gamma_\mu \gamma_a \tag{25}$$

Now using the fact that the suffix 'a' can always be adjusted to coincide with a particular value of the dummy suffix λ or any suffix 'a'' which like 'a' is not dummy, we obtain

$$\gamma_\mu \gamma_\nu U \gamma_\mu \gamma_\nu = 4 \gamma_\mu A_2 A_3 \ldots A_n \gamma_\mu A_1 \tag{26}$$

$$= 4 \gamma_\mu \gamma_a \gamma_{a'} A_2 A_3 \ldots A_n \gamma_\mu \gamma_a \gamma_{a'} A_1 \tag{27}$$

As stated before throughout this paper the repeated suffix 'a' does not imply any summation. Repeatedly applying the procedure adopted in the derivation of equation (26) with the help of equation (24), and using relation (27), we obtain the following relation for an even string U.

$$\gamma_\mu \gamma_\nu U \gamma_\mu \gamma_\nu = 4 \gamma_\mu \gamma_a A_3 A_4 \ldots A_n \gamma_\mu \gamma_a A_2 A_1 \tag{28}$$

$$= \gamma_\mu \gamma_\nu A_{r+1} A_{r+2} \ldots A_n \gamma_\mu \gamma_\nu A_r A_{r-1} \ldots A_1 \tag{29}$$

$$= 4 \gamma_\mu A_r A_{r+1} \ldots A_n \gamma_\mu A_{r-1} A_{r-2} \ldots A_1 \tag{30}$$

$$= -8 U_R \tag{31}$$

where r is even in equations (29) and (30).

From relations (26) and (31) we get the following results for an odd string S

$$\gamma_\mu S \gamma_\mu = -2 S_R \tag{32}$$

From equations (24) and (31) we find

$$\gamma_\mu U \gamma_\mu = 2\gamma_a [U + U_R] \gamma_a \tag{33}$$

For a string V' occurring within the trace bracket we have the following relations similar to those given by equations (22) and (23)

$$\tfrac{1}{2}[\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu]([\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu] V') = [\gamma_\mu \gamma_a - \gamma_a \gamma_\mu]([\gamma_\mu \gamma_a - \gamma_a \gamma_\mu] V') + \gamma_5 [\gamma_\mu \gamma_a - \gamma_a \gamma_\mu](\gamma_5 [\gamma_\mu \gamma_a - \gamma_a \gamma_\mu] V') \tag{34}$$

$$= \tfrac{1}{4}[\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu]([\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu] V') + \tfrac{1}{4}\gamma_5 [\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu](\gamma_5 [\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu] V') \tag{35}$$

Equations (34) and (35) yield the following relations

$$\gamma_5 \gamma_\mu \gamma_\nu (\gamma_5 \gamma_\mu \gamma_\nu V') = \gamma_\mu \gamma_\nu (\gamma_\mu \gamma_\nu V') + 4(V') - 4\gamma_5(\gamma_5 V') \tag{36}$$

and

$$\gamma_\mu \gamma_\nu (\gamma_\mu \gamma_\nu V') + \gamma_5 \gamma_\mu \gamma_\nu (\gamma_5 \gamma_\mu \gamma_\nu V') = 4\gamma_\mu \gamma_a (\gamma_\mu \gamma_a V') + 4\gamma_5 \gamma_\mu \gamma_a (\gamma_5 \gamma_\mu \gamma_a V') \tag{37}$$

Let us transfer in equation (37) the elements of V' $(= A_1' A_2' \ldots A_n')$ from the inside of the trace bracket to the outside following the procedure used in obtaining equations (25) to (31). This enables us to derive from equation (37) the following relation

$$\gamma_\mu \gamma_\nu (\gamma_\mu \gamma_\nu V') + \gamma_5 \gamma_\mu \gamma_\nu (\gamma_5 \gamma_\mu \gamma_\nu V') = 16 V_R' \tag{38}$$

Putting $U = V_R'$ in equation (38) we obtain the following expression for U.

$$16U = 2\gamma_\mu \gamma_\nu (\gamma_\mu \gamma_\nu U_R) + 4\gamma_5(\gamma_5 U) - 4(U) \tag{39}$$

Equations (39) and (11) also lead to the relation

$$2U + 2U_R = (U) + \gamma_5(\gamma_5 U) \tag{40}$$

Using relation (40) we obtain from equation (33)

$$\gamma_\mu U \gamma_\mu = (U) - \gamma_5(\gamma_5 U) \tag{41}$$

$$= 2(U) - 2U - 2U_R \tag{42}$$

Putting $V' = \gamma_a S$ (where S is an odd string) in equations (37) and (38), we obtain the following relation

$$4S = \gamma_\mu(\gamma_\mu S_R) + \gamma_5 \gamma_\mu (\gamma_5 \gamma_\mu S_R)$$
$$= \gamma_\mu(\gamma_\mu S) - \gamma_5 \gamma_\mu (\gamma_5 \gamma_\mu S) \tag{43}$$

Equation (43) provides us with the following relation

$$(S' \gamma_\mu)(\gamma_\mu S) = 2([S + S_R] S') \tag{44}$$

Equations (32) and (44) were already deduced by Chisholm (1963) in a different manner.

Equation (44) leads to the relation

$$(S\gamma_\mu[1 + \gamma_5])(S' \gamma_\mu [1 + \gamma_5]) = 4(S_R S'[1 - \gamma_5]) \tag{45}$$

From equations (42) and (33), with $\gamma_a S$ replacing U, we obtain, after some manipulation, the following result

$$(\gamma_a S)(\gamma_a S') = ([S_R + S][S' + \gamma_a S' \gamma_a]) \tag{46}$$

The substitution, $U = \gamma_a S$ in equation (40) leads to the following relation

$$\gamma_a S \gamma_a = -S_R + \tfrac{1}{2}\gamma_a(\gamma_a S) + \tfrac{1}{2}\gamma_5 \gamma_a(\gamma_5 \gamma_a S) \tag{47}$$

From equations (39) and (40) we have

$$(\gamma_\mu \gamma_\nu U)(\gamma_\mu \gamma_\nu V) = 4([U_R - U] V) + 4(U)(V) \tag{48}$$

and

$$(\gamma_\mu \gamma_\nu U[1 + \gamma_5])(\gamma_\mu \gamma_\nu V[1 + \gamma_5]) = -16(UV[1 + \gamma_5]) + 8(U[1 + \gamma_5])(V[1 + \gamma_5]) \tag{49}$$

Equation (40) leads to the relation

$$(\gamma_5 U)(\gamma_5 V) = 2(V[U + U_R]) - (U)(V) \tag{50}$$

we may note that the r.h.s. of equation (50) does not involve the γ_5 matrix. From equation (50) we have

$$([1 + \gamma_5] U)([1 + \gamma_5] V) = 2(V[U + U_R][1 + \gamma_5]) \tag{51}$$

Taking $S = A_1 A_2 \ldots A_m$ and $S' = A_{m+1} A_{m+2} \ldots A_n$ and using equation (43) we obtain the following formula for determining $(\gamma_5 U)$,

$$4(\gamma_5 U) = 4(\gamma_5 A_1 A_2 \ldots A_n) = -\sum_{r=1}^{m} (-1)^r (A_1 A_2 \ldots A_{r-1} A_{r+1} \ldots A_m) . (\gamma_5 A_r S') - \sum_{t>m}^{n} (-1)^t (\gamma_5 A_t S)(A_{m+1} A_{m+2} \ldots A_{t-1} A_{t+1} \ldots A_n) \tag{52}$$

From the relation

$$2(\gamma_5 [U + U_R] V) = (U)(\gamma_5 V) + (\gamma_5 U)(V) \tag{53}$$

obtained from equation (40), we can develop another efficient method for determining $(\gamma_5 A_1 A_2 \ldots A_n)$.

Putting $U = A_1 A_2 A_3 A_4$ and $V = A_5 A_6 \ldots A_n$ in equation (53) we get the following relation

$$4(\gamma_5 A_1 A_2 \ldots A_n) = 4(\gamma_5 UV) = 2(\gamma_5 [U - U_R + (U)] V) - (\gamma_5 V)(U) + (V)(\gamma_5 U) \tag{54}$$

where

$$\tfrac{1}{2}[U - U_R + (U)] = A_1 . A_2 A_3 A_4 - A_1 . A_3 A_2 A_4 + A_1 . A_4 A_2 A_3 + A_2 . A_3 A_1 A_4 - A_2 . A_4 A_1 A_3 + A_3 . A_4 A_1 A_2 \tag{55}$$

Equations (52), for the case $m = 3$, and (54) have also been derived in our previous paper (Sarkar, 1971) in a different manner.

Replacing V by $V\gamma_5$ in equation (54) we can obtain an efficient method for determining $(A_1 A_2 \ldots A_n)$.

From equation (54) we have

$$4([1+\gamma_5]A_1 A_2 \ldots A_n) = 2([1+\gamma_5][U - U_R + (U)]V) - ([1-\gamma_5]U)([1+\gamma_5]V) \tag{56}$$

Let us now obtain some results involving four 2×2 matrices, viz., σ_i and σ_4 (which is defined by equation (9)) with the help of the formulae involving the Dirac matrices already derived in this paper. Using equations (24) (for $a=4$ and $a=1$) and (31) we can obtain with the help of relations (4) to (10) the following results:

$$\sigma_\mu u \sigma_\mu = -\sigma_1 \sigma_\mu u \sigma_1 \sigma_\mu \tag{57}$$

$$= -\tfrac{1}{4}\sigma_\mu \sigma_\nu u \sigma_\mu \sigma_\nu \tag{58}$$

$$= 2\bar{u}_R \tag{59}$$

where the form of u is given in equation (7). The form of $\bar{u}_R$ is obtained with the help of equations (17) and (19).

From equation (39) we obtain

$$4U\,[1+\gamma_5] = \gamma_\mu \gamma_a [1+\gamma_5](\gamma_\mu \gamma_a U_R[1+\gamma_5]) \tag{60}$$

$$= \tfrac{1}{4}\gamma_\mu \gamma_\nu [1+\gamma_5](\gamma_\mu \gamma_\nu U_R[1+\gamma_5]) \tag{61}$$

With the assistance of equations (4) to (7) we can obtain from equations (60) and (61) the following relations

$$4u = 2\sigma_\mu \sigma_1(\sigma_\mu \sigma_1 \bar{u}_R) \tag{62}$$

$$= -2\sigma_\mu(\sigma_\mu \bar{u}_R) \tag{63}$$

$$= \tfrac{1}{2}\sigma_\mu \sigma_\nu(\sigma_\mu \sigma_\nu \bar{u}_R) \tag{64}$$

From equations (63) and (9) we have

$$2(uv) = (\bar{u}_R)(v) - (\sigma_i v)(\sigma_i \bar{u}_R) \tag{65}$$

From the relation

$$(U[1+\gamma_5]).[1+\gamma_5] = 2[U_R + U][1+\gamma_5] \tag{66'}$$

derived from equation (40) and using equations (8), (5) and (6) we obtain the following result

$$(u) = u + \bar{u}_R \tag{66}$$

Equation (63) combined with equation (66) leads to the relation

$$(\sigma_i v)(\sigma_i u) = ([u - \bar{u}_R]v) \tag{67}$$

From equation (59) we get the relation

$$\sigma_i u \sigma_i = u + 2\bar{u}_R \tag{68}$$

From equations (58) and (64) it is possible to evaluate $\sigma_i \sigma_j u \sigma_i \sigma_j$ and $(\sigma_i \sigma_j u)(\sigma_i \sigma_j v)$.

Replacing u by $\sigma_a u \sigma_4$ $(= i\sigma_a u)$ in equation (66) and then multiplying the resulting equation by σ_a, from the right we obtain the following relation

$$\sigma_a u \sigma_a = \sigma_a(\sigma_a u) + \bar{u}_R \tag{69}$$

Equation (69) leads to the relation

$$(\sigma_a v)(\sigma_a u) = (\sigma_a u \sigma_a v) - (\bar{u}_R v) \tag{70}$$

In equations (69) and (70) the repeated index 'a' does not indicate any summation and can assume any specific value from 1 to 3.

With the help of equations (8), (56), (55), (10), (14) and (15) we obtain the following reduction formula for the trace (u)

$$\begin{aligned}(u) = (a_1 a_2 \ldots a_n) = \tfrac{1}{2}(U[1+\gamma_5]) = ([a_1 . a_2 a_3 a_4 - a_1 . a_3 \bar{a}_2 a_4 \\ + a_1 . a_4 \bar{a}_2 \bar{a}_3 + a_2 . a_3 a_1 a_4 - a_2 . a_4 a_1 \bar{a}_3 \\ + a_3 . a_4 a_1 a_2] a_5 a_6 \ldots a_n) - \tfrac{1}{2}(a_4 a_3 a_2 a_1)(a_5 a_6 \ldots a_n)\end{aligned} \tag{71}$$

wherein $\bar{a}_r$ has been defined in equation (19).

We may note that the 4×4 Dirac matrices γ_μ generate a sixteen-element algebra—the Dirac algebra—of which the elements are 1, γ_μ, $\sigma_{\mu\nu}$, $i\gamma_\mu\gamma_5$ and γ_5, whereas the 2×2 Pauli matrices σ_i generate a four-element algebra, of which the elements are σ_i and 1. It may be convenient for numerical calculation of the trace of 4×4 Dirac matrices if we reduce it to one involving 2×2 matrices σ_μ with the help of the formulae derived in this paper.

Acknowledgement

The author wishes to thank Prof. N. C. Sil for taking kind interest in this paper.

References

Chisholm, J. S. R. (1963). *Nuovo Cimento*, **30**, 426.
Chisholm, J. S. R. (1966). *Nuovo Cimento*, **43A**, 1185.
Sarkar, S. (1971). *Acta Physica Academiae Scientiarum Hungaricae*, **30**, 351.